AN ENUMERATION OF THE REAL NUMBERS
(UMA ENUMERAÇÃO DOS NÚMEROS REAIS)

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ABSTRACT

The presented theory outlines a method for representing both rational and irrational real numbers within the interval $[0, 1)$ using a balanced binary tree structure. This approach enables the calculation of rational and irrational numbers with infinitely increasing degrees of precision, permits the enumeration of all real numbers in the interval $[0, 1)$, and, ultimately, across the entire real line. This could potentially challenge Cantor's theory. Furthermore, the theory concludes that rational and irrational numbers are interleaved in their distribution along the geometric real line, analogous to the distribution of even and odd integers.

***K*eywords** Cantor · Irrational Numbers · Rational Numbers · Enumeration · Example

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Conflict of Interest

The author of this manuscript affirms that there are no conflicts of interest to disclose regarding the publication of this article. The author has no financial or personal relationships that could inappropriately influence or bias the work reported in this manuscript.

1 Introduction

In this study, building upon prior work by the author [9], I delve into additional specifics regarding the proof that: (i) the real numbers within the interval $[0, 1)$ are enumerable; (ii) the sets of rational and irrational numbers in $[0, 1)$ possess the same cardinality; (iii) the entirety of the real numbers set is also enumerable; (iv) there is a regular pattern in the distribution of rational and irrational numbers along the real line. Additionally, (v) functional examples are provided for the enumeration of real numbers within the interval $[0, 1)$ and, also, (vi) of all real numbers.

2 Methodology

In the present study, I use the same methodology used in [9], which I repeat here, with minor adaptations related to the references: "traditional Mathematics methods and some Philosophy methods. According to [7] and [4], the methods of Philosophy are 'the methods of reasoning and analysis that seek to clearly define the concepts used, investigate and expose the foundations of ideas and theories and build a systematic theory that is based on other ideas and systems of thought'.

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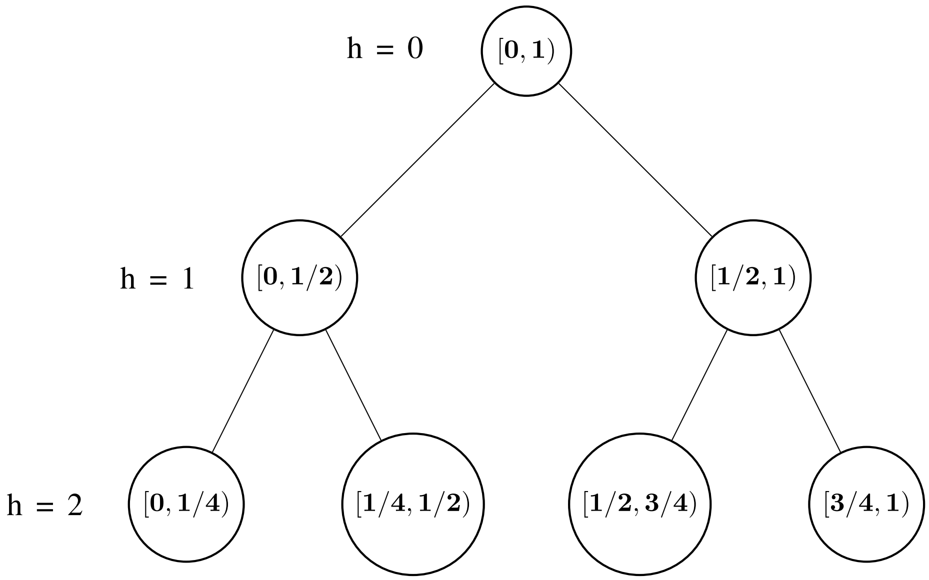

Figure 1: Balanced Binary Tree. Infinite Subdivision of the Interval [0,1)

In this context, it is essential to precisely define concepts, critique ideas, and compare them, emphasizing their limitations or strengths [4, 177]. Similarly, strive to uncover and elucidate the underlying principles of ideas, thereby laying the groundwork for a thorough critique. Essential to this study is the development of an integrated system that provides explanatory and insightful value. Hence, this study aligns with the positivist school of thought, employing 'logic, reason, rigor, and inference in the pursuit of knowledge' [4, 189]. As per [7], 'it also incorporates both deduction and induction, thereby drawing both *necessary* and *probable* conclusions [4, 190], aiming to both challenge an existing theory and formulate a new one.'"

3 Results and Discussion

Consider the binary tree that represents the half-open interval on the Real line $[0, 1)$ as depicted in Figure 1 (See [8, 9]).

The number of leaf nodes for each height h ($h = 0, 1, 2, ..., n, ...$) of the tree is 2^h and the lengths of the sub-intervals represented by each such node is 2^{-h}.

The extremes of the sub-intervals represented by all the leaf nodes at height h if the set $Rac(h) = \{(0/2^{-h}) = 0,\ (1/2^{-h}),\ (2/2^{-h}),\ ...,\ (2^{-h} - 1)/2^{-h},\ (2^{-h}/2^{-h}) = 1\}$. It is obvious that all the numbers in this set are rational numbers. Additionally, they are all the rational numbers in $[0, 1)$ with h binary digits (0 or 1) after the dot (.). To see this, consider, first, the following excerpt from article [8]:

> Another aspect to verify initially is that all the boundaries of the sub-intervals denoted by the tree nodes are rational numbers. Let's explore how we can depict each of these boundaries, in other words, each of these rational numbers, utilizing a binary representation of fractional values. Consider, first, the height $h = 0$. At this stage, the tree solely consists of the root node with boundaries at 0 and 1. These boundaries, these rational numbers, are depicted by the non-fractional binary numbers $(0.)$ and $(1.)$. Let's proceed to height $h = 1$. At this point, we have two sub-intervals whose endpoints are 0, and 1. These boundaries, or rational numbers, can be expressed in binary as 0.0, 0.1, and 1.0, signifying $0 + 0 * (1/2)^{-1} = 0$, $0 + 1 * (1/2)^{-1} = 1/2$ and $1 + 0 * (1/2)^{-1} = 1$. Let us now consider height $h = 2$. At this point, we have four sub-intervals with endpoints at 0, $1/4$, $1/2$, $3/4$, and 1. These rational numbers can be represented by the binary numbers 0.00, 0.01, 0.10, 0.11 and 1.00, that is, $0+0*(1/2)^{-1}+0*(1/2)^{-2} = 0$, $0+0*(1/2)^{-1}+1*(1/2)^{-2} = 1/4$, $0+1*(1/2)^{-1}+0*(1/2)^{-2} = 1/2$, $0 + 1 * (1/2)^{-1} + 1 * (1/2)^{-2} = 3/4$, $1 + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} = 1$.

Generally, at height h, the boundaries of the sub-intervals of the leaf nodes at height h compose the set $Rac(h) =$:

$$\mathbf{0} + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} + \ldots + 0 * (1/2)^{-(h-1)} + 0 * (1/2)^{-h},$$
$$\mathbf{0} + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} + \ldots + 0 * (1/2)^{-(h-1)} + \mathbf{1} * (1/2)^{-h},$$
$$\mathbf{0} + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} + \ldots + \mathbf{1} * (1/2)^{-(h-1)} + 0 * (1/2)^{-h},$$
$$\mathbf{0} + 0 * (1/2)^{-1} + 0 * (1/2)^{-2} + \ldots + \mathbf{1} * (1/2)^{-(h-1)} + \mathbf{1} * (1/2)^{-h},$$

$$, \ldots,$$

$$\mathbf{0} + \mathbf{1} * (1/2)^{-1} + \mathbf{1} * (1/2)^{-2} + \ldots + \mathbf{1} * (1/2)^{-(h-1)} + \mathbf{1} * (1/2)^{-h}.$$

Another representation for the set $Rac(h)$, for any $h = \{1, 2, 3, \ldots\}$ is:

$$(0 = 0/2^h),\ 1/2^h,\ 2/2^h,\ 3/2^h,\ \ldots,\ (2^h - 1)/2^h,\ (2^h/2^h = 1).$$

All these numbers are rational numbers. Consider now, any rational number, which can always be represented in **decimal** form, as is widely known. Specifically, any integer number can be represented in decimal form, and there is a widely known algorithm to divide two integers in decimal form to produce a decimal representation of the original rational number. As any integer in decimal form can be easily converted into **binary** form by successive divisions by 2, and there is a simple algorithm to divide two integer numbers represented in binary form, resulting in a binary representation of the result, any rational number in decimal notation can be converted into binary notation this way. Another way to see this is to separate the whole and the fractional parts of any rational number in decimal notation. The whole part can be easily converted into binary notation by successive divisions by 2. As for the fractional part, it can also be easily converted into binary notation by successive multiplications by 2. Since both the whole and fractional parts of any rational number in decimal notation can be converted into binary form, any rational number can be represented in binary form. It is also widely known that all rational numbers in binary notation have a finite or finitely repeating pattern of binary digits. This is a consequence of the division algorithm of numbers in any base (b). As the base is finite, the remainder will eventually turn into 0, which means that the representation is finite, or will repeat itself after at most b steps, leading to an infinitely repeating pattern of the digits of the rational number being computed.

We can conclude that any rational number in $[0, 1)$ will eventually appear as an extreme of one of the leaf nodes for a height h, i.e., a number belonging to the set $Rac(h)$, or will have its binary digits defined, by any height h, by the digits of a number in the set $Rac(h)$, so that, when the tree grows, i.e., $h \to \infty$, these binary digits will form an infinitely repeating pattern of binary digits, in a zig-zap fashion down the tree, obtained successively from the digits of increasingly precise numbers from $\{Rac(1), Rac(2), Rac(3), \ldots, Rac(h), \ldots\}$.

One important point is provided by the following excerpt from article [9]:

> For any height h of the tree, and excluding the left and right ends of the sub-intervals represented by the leaves of the tree, which are all rational numbers, the interior of any of these intervals may be represented by $S_i = (l_i, r_i)$, where l_i and r_i are rational numbers, and their union by $\bigcup_{i=1}^{2^h} S_i = S_1 \cup S_2 \cup \ldots \cup S_{2^h} = (l_1, r_1) \cup (l_2, r_2) \cup \ldots \cup (l_{2^h}, r_{2^h})$.

As $h \to \infty$, any irrational number $irrational \in [0, 1)$ will be such that $irrational \in S_i$ for $i = 1, 2, 3, \ldots, 2^h$, for one an only one such i for any height h. The length of any S_i for $i = 1, 2, 3, \ldots, 2^h$ tends to 0 as $h \to \infty$. Therefore, the quantity of real numbers in any such S_i, $Q_i(S_i)$, is such that $Q_i(S_i) \to 1$ as $h \to \infty$. If it would be at least two (2), there would be a positive distance d between these two numbers, which is contradictory to the fact that the length of any such $S_i \to 0$, as $h \to \infty$. So, $Q_i(S_i) \to 0$ or $Q_i(S_i) \to 1$, as $h \to \infty$. The crucial observation is that, in the limit (∞), $Q_i(S_i) \to 0$, but this limit is never reached, which means that there will alway be one real number in any such S_i, for any height h. And only one, as there cannot be 2 or more. And this real number is irrational, since all the rational numbers, as already proved, eventually appear as a number belonging to the set $Rac(h)$, i.e., outside S_i, for a sufficiently large h, or obtain their digits from the digits of increasingly precise numbers from $\{Rac(1), Rac(2), Rac(3), \ldots, Rac(h), \ldots\}$, therefore, also outside any S_i for all heights $h_b < h$ as $h \to \infty$.

We can conclude that the number of rationals $QRac\,[0, 1) = \lim_{h\to\infty} 2^h = QIrrac\,[0, 1) = \lim_{h\to\infty} 2^h$, as 2^h is the number of leaf nodes of the tree for any height h. The rational numbers in $[0, 1)$ would be the extremes of the intervals in $\bigcup_{i=1}^{2^h} S_i = S_1 \cup S_2 \cup \ldots \cup S_{2^h} = (l_1, r_1) \cup (l_2, r_2) \cup \ldots \cup (l_{2^h}, r_{2^h})$, as $h \to \infty$, excluding the number 1. The irrational numbers in $[0, 1)$ would the unattainable theoretical limits that remain inside the intervals in

$\bigcup_{i=1}^{2^h} S_i = S_1 \cup S_2 \cup ... \cup S_{2^h}$, one and only one irrational number in each of the intervals, as $h \to \infty$. The rational and irrational numbers in $[0, 1)$ are, therefore, interleaved, as, by analogy, the even and odd natural numbers.

If we represent any sub-interval $S_i = (l_{(h,i)}, r_{(h,i)})$, for any leaf node at height h, such that $l_{(h,i)} < r_{(h,i)}$, as shown in [9], "$\mid r_{(h,i)} - l_{(h,i)} \mid > 0$, for any h, and $\mid r_{(h,i)} - l_{(h,i)} \mid \to 0$ as $h \to \infty$. Therefore $l_{(h,i)} \to r_{(h,i)}$ and $r_{(h,i)} \to l_{(h,i)}$ as $h \to \infty$". As a consequence, all the irrational numbers in $[0, 1)$ can be approximated, with infinitely increasing precision, by two sequences of rational numbers, for, as already stated in [9], "any supposedly irrational number" $irrational \in [0, 1)$, "at the root node of the tree", "belongs to an infinite series of nested sub-intervals down the tree $\{(l_1 = 0, r_1 = 1), (l_2, r_2), (l_3, r_3)...\}$, with $(0 = l_1) \leq l_2 \leq l_3 \leq ... < irrational < ... \leq r_3 \leq r_2 \leq (r_1 = 1)$, whose sizes tend to 0 as $h \to \infty$".

Consider, now, that the number of leaf nodes for any height h of the tree is 2^h, so that the quantity of rational numbers plus the quantity of irrational numbers "represented" at height h of the tree, i.e., the extremes of the intervals S_i, excluding the number 1, representing the quantity of rational numbers obtained and represented by the tree at height h, plus the number of the interiors of the intervals S_i, which would represent the irrational numbers that remain in the construction of the tree at height h, is $QReal(h) = 2^{h+1}$. So that the quantity of real numbers represented in the tree for all heights up to height $h-1$, with duplicity, is, therefore, $QReal(0, \ldots, h-1) = (\sum_{i=1}^{h} 2^i)$.

Define the function $E_1(h = 0, -1, rational = 0) = 0$ and $E_1(h = 0, -1, irrational) = 1$ and, for any height $h = \{1, 2, 3, \ldots\}$ and $n = 0, 1, 2, \ldots, 2^h - 1$, $E_1(h, n, rational = n/2^h) = (\sum_{i=1}^{h} 2^i) + (2 * n)$, for the rational numbers in $[0, 1)$ represented at height h; and $E_1(h, n, (n/2^h, irrational)) = (\sum_{i=1}^{h} 2^i) + (2 * n) + 1$, for the irrational numbers represented at height h, is an enumeration of all the the rational and irrational numbers represented in the tree for all the heights up to and including height h.

For any height $h = \{0, 1, 2, 3, \ldots\}$ and $n = 0, 1, 2, \ldots, 2^h - 1$, the function $E_2(h, n/2^h, rational) = 2 * n$ and $E_2(h, n/2^h, irrational) = 2 * n + 1$ is an enumeration of all the rational and irrational numbers represented in the tree at height h. By making $h \to \infty$, we can make $E_2(h)$ an enumeration of all the real numbers, rational and irrational, in the interval $[0, 1)$.

Therefore, according to this theory, all the real numbers in the interval $[0, 1)$ can be enumerated, with the above functions E_1 and E_2, making $h \to \infty$. Obviously, both E_1 and E_2 can be considered their inverses, mapping the natural numbers $\{0, 1, 2, \ldots\}$ to the real interval $[0, 1)$. Given an enumeration of $[0, 1)$, $E_{[0,1)}(n) : n \to x \in [0, 1)$, we can build an enumeration of $[0, \infty)$ by $E_{[0,+\infty)}(n) : n \to E_{[0,1)}(n)/(1 - E_{[0,1)}(n))$, where $n = \{0, 1, 2, \ldots\}$. And, given an enumeration of $[0, +\infty)$, $E_{[0,+\infty)}(n) : n \to x \in [0, +\infty)$, where $n = \{0, 1, 2, \ldots\}$, we can build an enumeration of $(-\infty, +\infty)$, $E_{Real}(n) : n \to \ln(E_{[0,+\infty)}(n))$.

Alternatively, we could build an enumeration $E_{(-x,x)}$, for any $x \in \mathbb{R}$, with the construction of a similar binary tree for $(-x, x)$, as in Figure 1, and create an enumeration $E_{(-\infty,+\infty)} = \lim_{x \to \infty} E_{(-x,x)}$.

Finally, as already cited in [9]:

> In the practical technique of measurement in the physical-material world, there would no be irrational numbers, but only approximations by rational numbers with different degrees of precision. In Geometry, however, which is a theoretical science, the irrational numbers do exist, as is the case of the number $\sqrt{2}$, the size of the diagonal of a square with sides of length 1. It is impossible to measure exactly this diagonal in practice, but the theoretical limit, in Geometry, can be calculated, and is the irrational number $\sqrt{2}$. Likewise, in any physical representation of the Real line, the approximations of rational numbers with increasing precision would leave no wholes to be filled, but gaps, whose lengths would tend to 0. In a geometrical Real line, however, there would also be such gaps whose lengths tend to 0, but, as there is no limit (or cap) to the precision that can be attained in the theory of Geometry, we could imagine pairs of rational numbers, as in the leaves of the depicted tree, tending to each other with infinite precision, but never reaching one another, so that, in the limit, there would exist a point between any such pair of rational numbers (and only one, as they tend to each other), representing an irrational number, that would surely exist.

4 Final Considerations

Building upon the concepts previously established in [8] and [9], this work presents additional insights that bolster the theory proposing the enumerability of rational and irrational numbers within the interval $[0, 1)$ and, ultimately, throughout the entire real line, potentially presenting a contradiction to Cantor's theory. Moreover, I introduce specific enumeration functions for real numbers in the intervals $[0, 1)$, $[0, +\infty)$, and the entire real line. In conclusion, the findings suggest that rational and irrational numbers are evenly distributed along the real line, interleaved in a manner analogous to that of even and odd integers.

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